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# Using periodic modulation to control coexisting attractors induced by delayed feedback

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**Abstract**

A delay in feedback can stabilize simultaneously several unstable periodic orbits embedded in a chaotic attractor. We show that by modulating the feedback variable it is possible to lock one of these states and eliminate other coexisting periodic attractors. The method is demonstrated with both a logistic map and a $CO_2$ laser model.


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## 1. Introduction

In the past decade chaos control has attracted much attention and many techniques for controlling chaotic dynamics have been developed (see, for example, [1] and references therein). Among a number of methods which use a feedback loop, one may distinguish the two most extensively employed methods. These are the method of Ott, Grebogi, and Yorke [2] (OGY) based on small perturbations to a system parameter to stabilize a given periodic orbit embedded in a chaotic attractor and the method of Pyragas [3] who proposed to modify state variables instead of control parameters. The main contribution of Pyragas was to introduce a correction based on delayed variable as stated later rather than action on a variable. The correction signal is proportional to the difference between values of a given variable at different times; the delay time is selected equal to the period of the unstable periodic orbit (UPO) to be stabilized. All the methods propose to act on the state variable, whether this is directly or indirectly (through action on some parameter). Moreover, many methods, including variants of the OGY method, do not require detailed knowledge of the system, otherwise they would not have been implemented experimentally so rapidly in systems mentioned in [4–11]; the knowledge on the system necessary to select the perturbation can be obtained by simply observing the system for a suitable learning time. An advantage of the Pyragas method is that it does not require full information about the target UPO; but it only uses a constant delay in time in the feedback loop. In practice, the Pyragas method transforms a system of ordinary

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differential equation into a delayed dynamical system. This implies to increase its dimensionality so as the desired unstable periodic orbit, which was unstable in the ordinary differential equation system, becomes now stable in the new delayed dynamical system.

A time delayed feedback is known to be an efficient tool for continuous-time control of dynamical systems. However, there is a problem that the delayed feedback can induce multiple coexisting attractors. This phenomenon predicted by Ikeda et al. [12,13] was confirmed experimentally in an electro-optical bistable device with a computer delay [14]. Later, a very large number of multistable self-oscillatory modes were observed in a laser diode pumped hybrid bistable system with a large delay [15]. The main difference between the cases of large and small delay is the variety of coexisting attractors. When the delay time is much larger than the response time, there is a fundamental oscillation mode which exhibits period-doubling bifurcations and there are also multiple harmonic oscillation modes, each of which exhibits a sequence of bifurcations [12].

In practical applications, when one of the attractors according to some criteria would yield superior system performance over the others, it is important to be able to drive most trajectories to the desirable attractor rapidly by using only small feedback control to an accessible parameter or state of the system. It is possible to provide some regulation of a steady state of the nonlinear system through an adaptive control mechanism [16,17], which utilizes an error signal proportional to the difference between the goal output and the actual trajectory. This error signal governs a change of the system parameters reducing the error to zero. Another control algorithm to drive trajectories to a desirable attractor by using small feedback control, has been suggested by Lai [18]. His idea is to build a bush-like structure of paths to the target attractor and to stabilize a trajectory around one of the many paths, so that the trajectory asymptotically approaches the desirable attractor. However, at present, there is no guarantee that the method can be applied to practice. The success of the method relies on the region in the phase space to which the bush extends and the method is effective when there are fractal basin boundaries with large values of fractal dimension. In contrast, there is no appreciable increase in the probability for a trajectory to be driven to the desirable attractor if the basin boundaries are smooth.

There also exist other approaches in controlling systems with coexisting attractors. For instance, to add a harmonic modulation to an available system parameter [19]. This modulation with properly chosen frequency and amplitude can cause crisis of the attractors so that only a single attractor remains [20]. In other methods the coexisting attractors are not completely destroyed but their basins of attraction are changed so that a trajectory is attracted to one of the coexisting states. Among the latter methods, we should mention the method proposed by Pecora and Carroll [21]. They used chaotic signals with certain periodic characteristics as a part of the drive signal in a system with coexisting periodic attractors. Later, Yang et al. [22] modified this method, combining noise and a bias signal to achieve the same goal. Recently, Jiang [23] applied a periodic modulation to a system variable in order to drive a multistable system to a selected trajectory. However, in the all above mentioned works the systems were not chaotic and delayed feedback were not introduced.

In this Letter we will focus on the Pyragas method [3]. This method is based on the construction of a special form of time-continuous perturbation, which does not change the form of the desired UPO, but under certain conditions can stabilize it. Pyragas considered a system governed by the ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{Q}(\mathbf{x}, y), \qquad \dot{y} = P(\mathbf{x}, y) + F(t), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^{n-1}$ describes the variables of the system that are not available or not of interest and only $y \in \mathbb{R}$ can be measured as a system output, $\mathbf{Q}: \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ and $P: \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$, and $F(t)$ is the control signal. He proposed two types of feedback control, *delayed feedback control* [24]

$$F(t) = \eta\big[y(t-\tau) - y(t)\big] \tag{2}$$

and *external feedback control*

$$F(t) = \delta\big[\bar{y}(t) - y\big], \tag{3}$$

to stabilize an UPO embedded in the chaotic attractor. Here $\bar{y}(t)$ is the UPO to be stabilized and $\tau$ is its period. If the period of external force $T = 1/f$ or time delay $T_0$ is equal to $\tau$, then it is possible to find feedback strengths $\eta$ and $\delta$ which allow stabilization of the UPO. The delayed feedback control technique has been applied experimentally to many physical, chemical, and biological systems [1]. In particular, in laser physics it has been successfully used for stabilization of UPOs in a fiber laser [4], a $CO_2$ laser [5,6], a semiconductor laser [7], and an optically pumped far-infrared gas laser [8]. The efficiency of the delayed feedback method has been improved by the adaptive control technique [9] that was implemented experimentally with a $CO_2$ laser [10,11]. The external feedback control has been recently applied in experiments with an electronic circuit [25] and a periodically forced pendulum [26].

In this Letter we suggest to combine two types of the feedback control and apply both the delay and the periodic modulation. Our approach is illustrated schematically in Fig. 1. We consider a chaotic system with feedback and show that a time delay in feedback can stabilize simultaneously several UPOs which may coexist with the original chaotic attractor. We will show that the external control in the form of a harmonic modulation applied to the feedback strength allows locking one of these periodic orbits. Thus, the multistability can be controlled by modulating the feedback variable. As a result, only one of the coexisting attractors remains. Another important advantage of the oscillating feedback is that it allows one to overcome a limitation that occurs for the delayed feedback control. Following Giona's theorem [27], constant feedback can stabilize an UPO only if the stability matrix has no positive eigenvalues greater than unity. However, as was recently shown by Schuster and Stemmler [28], these restrictions can be overcome by modulating the feedback control periodically.

Our method will first be introduced for dynamical systems described by a map, then it will be extended to a system described by differential equations. In particular, we will demonstrate that homoclinic chaos

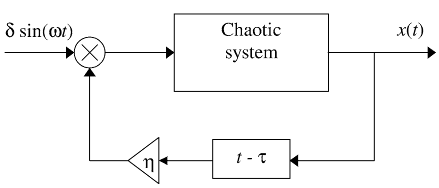


Fig. 1. Schematic of the method for controlling multistability in a time delayed feedback system.

in a $CO_2$ laser with feedback [29] can be stabilized to one of the periodic orbits. We choose the latter system because there are no, to our knowledge, efficient methods for controlling this type of chaos, although a phase locking of chaotic oscillations to external modulation has been proved recently [30,31].

The Letter is organized as follows. In Section 2 we apply our method to a chaotic logistic map. We show that the time delayed feedback induces bistability in this system, so that two periodic orbits are generated instead of chaos. Then, we add a harmonic modulation to the feedback strength to attract the trajectory to one of the attractors and eliminate the other. In Section 3 we demonstrate the applicability of our approach to a real physical model on an example of a $CO_2$ laser with feedback operating in the regime of homoclinic chaos. The delay in feedback stabilizes some periodic orbits which coexist with the original chaotic attractor and the feedback modulation locks one of the periodic states. Finally, conclusions are given in Section 4.

## 2. Application to the logistic map

### 2.1. Time delayed logistic map

The time delayed logistic map is one of the simplest systems which displays generalized bistability and can be described in the following form

$$x_{i+1} = ax_i(1 - x_i) - \eta(x_{i-k} - x_i), \tag{4}$$

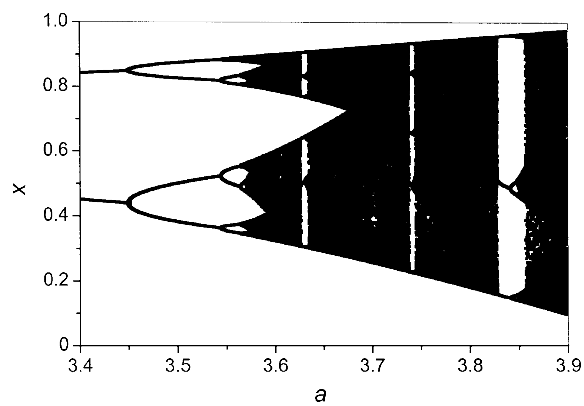


Fig. 2. Cascade of period-doubling bifurcations of the logistic map $x_{i+1} = ax_i(1 - x_i)$.

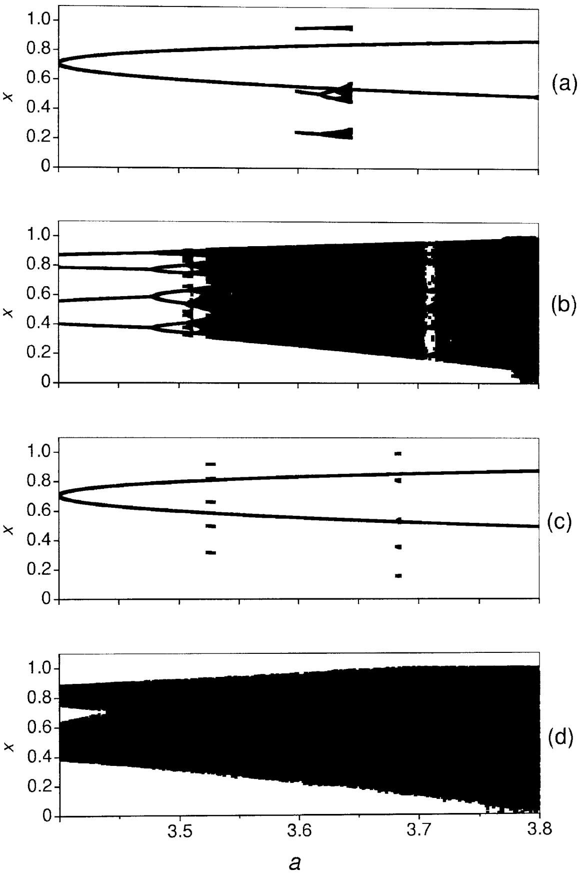


Fig. 3. Bifurcation diagrams of the delayed logistic map with respect to $a$ for $\eta = 0.19$ and delays (a) $k = 1$, (b) 2, (c) 3 and (d) 100.

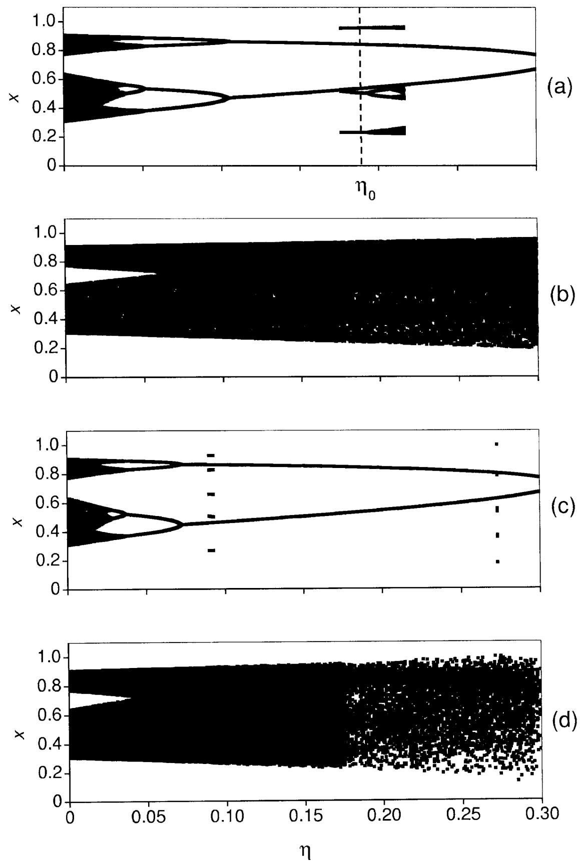


Fig. 4. Bifurcation diagrams with respect to $\eta$ for $a = 3.625$ and (a) $k = 1$, (b) 2, (c) 3 and (d) 100. The dashed line indicates the value of $\eta_0 = 0.19$ at which the control is applied.

where $x_{i+1}$ is measured as time series, with values in the interval [0,1], $a$ is the system parameter ($a \in [1, 4]$), $k$ is the time delay and $\eta$ is the feedback strength. The map Eq. (4) exhibits the coexistence of two attractors.

The bifurcation diagram of the logistic map Eq. (4) without feedback ($\eta = 0$) is shown in Fig. 2. In this Letter we explore the parameter range where the map exhibits chaos, close to the period-3 window. The delayed feedback changes drastically the system dynamics. These changes can be seen in Fig. 3 where we show four bifurcation diagrams with $a$ as a control parameter for different delays $k = 1, 2, 3, 100$ for fixed feedback strength $\eta = 0.19$. Short time delays induce generalized bistability in the system. When $k = 1$ (Fig. 3(a)) the whole bifurcation diagram is shifted towards higher values of $a$, while the period-3 window is shifted to lower values of $a$ and becomes wider, so that in the parameter range $a = [3.60, 3.65]$ two periodic attractors (period 2 and period 3) coexist instead of the chaotic attractor for the map without delay (Fig. 2). Generalized bistability is also observed with delays $k = 2$ (Fig. 3(b)) and $k = 3$ (Fig. 3(c)) but in narrower parameter ranges. However, in the map with a large delay ($k = 100$) bistability does not appear (Fig. 3(d)).

The coexistence of two attractors is also shown in the bifurcation diagrams with respect to feedback strength $\eta$ (Fig. 4). When $a$ is fixed in the chaotic region ($a = a_0 = 3.625$), generalized bistability appears in certain range of $\eta$ for $k = 1$ (Fig. 4(a)) and $k = 3$ (Fig. 4(c)), but not for $k = 2$ (Fig. 4(b)) and $k = 100$ (Fig. 4(d)). In the following we will consider only a

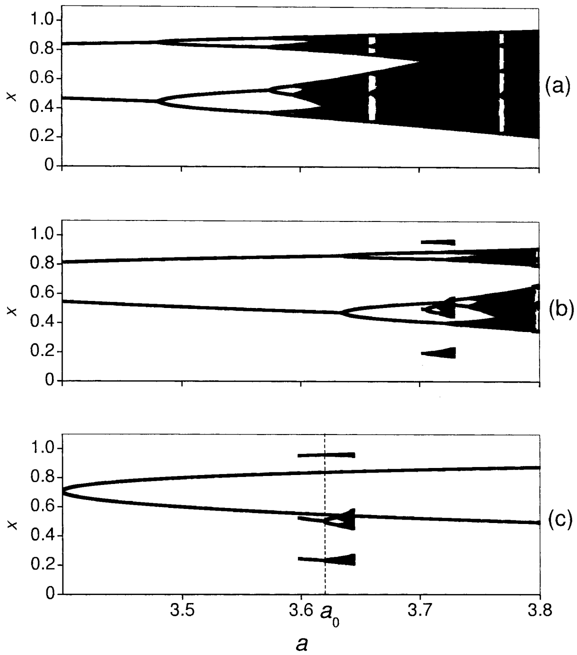


Fig. 5. Bifurcation diagrams for $k = 1$ and different feedback strengths (a) $\eta = 0.02$, (b) 0.11 and (c) 0.19. The dashed line indicates the initial value of $a_0 = 3.625$ at which the control is applied.

short delay ($k = 1$) because in this case the range of bistability is largest. The calculations show that the bistability range increases with the feedback strength. At small $\eta$ ($\eta = 0.02$ in Fig. 5(a)) the bistability range is almost not seen. However, with increasing $\eta$ the period-3 window becomes wider and is shifted to the lower values of $a$, while the whole diagram is shifted to larger $a$.

In the next subsection we will show that a periodic modulation to the system variable allows one to annihilate one of the coexisting attractors that results in monostability.

### 2.2. Modulation of the variable

The time delayed logistic map with modulation of the variable can be written in the following form

$$x_{i+1} = ax_i(1 - x_i) - \eta(x_{i-k} - x_i) - \delta \sin(2\pi f i), \quad (5)$$

where $\delta$ and $f$ are, respectively, the modulation amplitude and frequency. We will show that the weak ($\delta \ll 1$) and slow ($f \ll 1$) modulation can cause boundary crisis of one of the attractors, leading to its destruction. For instance, if we fix feedback strength $\eta$ on $\eta_0 = 0.19$ and parameter $a$ on $a_0 = 3.625$ in the range where the period-2 and period-3 attractors coexist, as shown in Figs. 4(a) and 5(c), and apply the control modulation with amplitude $\delta = 0.015$ and frequency $f = 0.02$ to the variable in the form of Eq. (5) at time $i = 100$, while the system stays initially in the period-3 attractor, the trajectory is attracted to the period 2 (Fig. 6). Thus, the period-3 attractor is annihilated by the modulation and the system becomes monostable. As seen from Fig. 6, the remaining period-2 state is amplitude modulated with the period of the control modulation, $1/f$. However, the amplitude of this modulation is relatively small because the modulation frequency is far from the resonant frequency of the period-2 attractor.

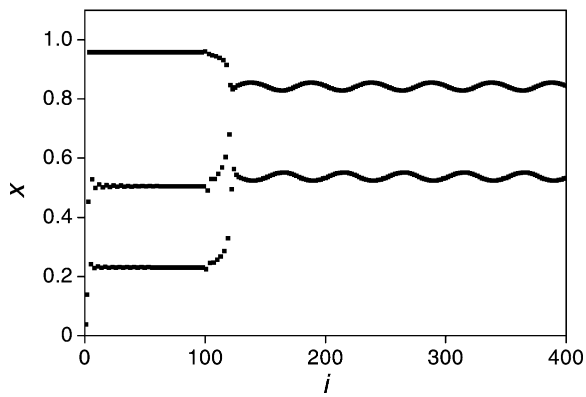


Fig. 6. Time series showing the annihilation the period-3 attractor when the control modulation with $\delta = 0.015$ and $f = 0.02$ is applied at time $i = 100$, $k = 1$, $a = 3.625$ and $\eta = 0.19$.

The physical mechanism underlying annihilation phenomenon in a system with parametrical modulation is described in [20]. This phenomenon results from boundary crisis due to a resonant interaction of the control frequency with the relaxation oscillation frequency of the associated attractor. The control modulation deforms the attractor and its basin of attraction [32] and may induce the period-doubling cascade of bifurcations terminated by chaos and crisis of the attractor. Similar mechanism is realized in the system with modulated feedback. The best conditions for the attractor annihilation is achieved when the control frequency $f$ is close to the relaxation oscillation fre-

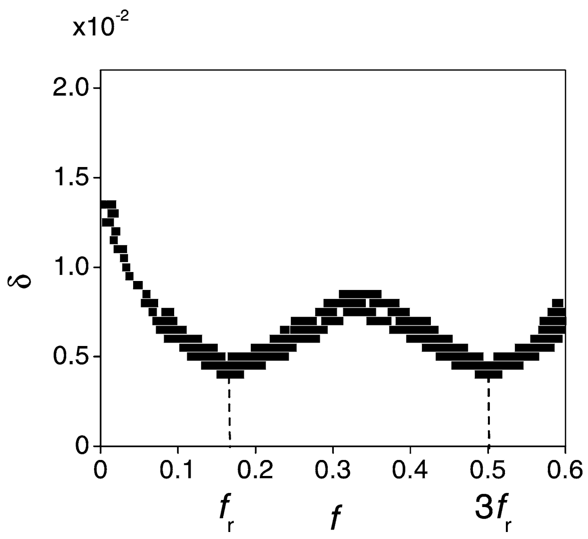


Fig. 7. Annihilation curve for modulation of the variable in the $(f, \delta)$ space for $k = 1$, $a = 3.625$ and $\eta = 0.19$. Below the curve the period-3 and period-2 attractors coexist while above the curve only the period 2 is stable.

quency $f_r$ of the attractor to be annihilated. We estimate this frequency from time series by measuring the dependence of the maximum response on the modulation frequency keeping the modulation amplitude fixed and small. At the resonant frequency the system response is maximal. For the logistic map Eq. (5) and our parameters, the relaxation oscillation frequency of the period-3 attractor $f_r \approx 0.168$.

The stability boundary of the period-3 attractor in the space of modulation parameters $f$ and $\delta$ is shown in Fig. 7. The dots in the figure form the annihilation curve. For the parameters below this curve, two attractors, period 2 and period 3, coexist, while above the curve only the period-2 attractor exists. The optimal conditions (minimal modulation amplitude) for the annihilation of the period-3 attractor are realized when $f = f_r$ and $f = 3f_r$. The later minimum can be understood from the following speculations. The period of relaxation oscillations of the period-3 attractor is a number of iteration points per period of the oscillations. The inverse period yields the relaxation oscillation frequency of the attractor. However, the period-3 attractor has three branches of points in time series (Fig. 6) and hence the number of iterations per period of relaxation oscillations in each branch, $i_1$, is a $1/3$ part of the total number of iterations per period for all three branches, $i_1 = i_r/3$. Thus the oscillation frequency for one branch $f_1 = 1/i_1 = 3f_r$ is also a resonant frequency for the period-3 attractor.

## 3. Application to a $CO_2$ laser

The specific property of the $CO_2$ laser with feedback is that it can displays chaos of Shilnikov type [29]. This chaotic motion is characterized by the erratic behavior when a parameter is varied towards the homoclinic condition associated with a saddle focus [33]. Normally Shilnikov chaos is located within very narrow parameter ranges and the chaotic windows are distributed exponentially with a control parameter [34,35]. Therefore, in experiments it is very difficult to fix the system in the neighborhood of a homoclinic orbit because of noise or small unavoidable fluctuations of the system parameters. Here we will show that Shilnikov chaos in a $CO_2$ laser with feedback can be stabilized by locking a particular unstable periodic orbit with a harmonic modulation applied to the feedback variable.

### 3.1. Model equations

The model is based on a four-level scheme for the active medium [36] leading to the following six-equation dynamical model

$$\dot{x}_1 = k_0 x_1 \{ x_2 - 1 - k_1 \sin^2 [\varepsilon x_6(\theta - T_0) + (1-\varepsilon) x_6 ] \}, \tag{6}$$

$$\dot{x}_2 = -\Gamma_1 x_2 - 2k_0 x_1 x_2 + \gamma x_3 + x_4 + P_0, \tag{7}$$

$$\dot{x}_3 = -\Gamma_1 x_3 + x_5 + \gamma x_2 + P_0, \tag{8}$$

$$\dot{x}_4 = -\Gamma_2 x_4 + \gamma x_5 + z x_2 + z P_0, \tag{9}$$

$$\dot{x}_5 = -\Gamma_2 x_5 + z x_3 + \gamma x_4 + z P_0, \tag{10}$$

$$\dot{x}_6 = -\beta x_6 + \beta B - \beta f(x_1), \tag{11}$$

where $f(x_1) = Rx_1/(1 + \alpha x_1)$ is the feedback function, $x_1$ is the normalized photon number proportional to the laser intensity, $x_2$ is proportional to the population inversion, $x_3$ is proportional to the sum of the populations on the two resonant levels, $x_4$ and $x_5$ are proportional, respectively, to difference and sum of the populations of the rotational manifolds coupled to the lasing levels. We assume that each manifold

Table 1
Parameter values used in simulations

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\Gamma_1$ | 10.0643 | $\alpha$ | 32.8767 | $k_0$ | 28.5714 | $\gamma$ | 0.05 |
| $\Gamma_2$ | 1.0643 | $\beta$ | 0.4286 | $k_1$ | 4.5556 | $P_0$ | 0.016 |

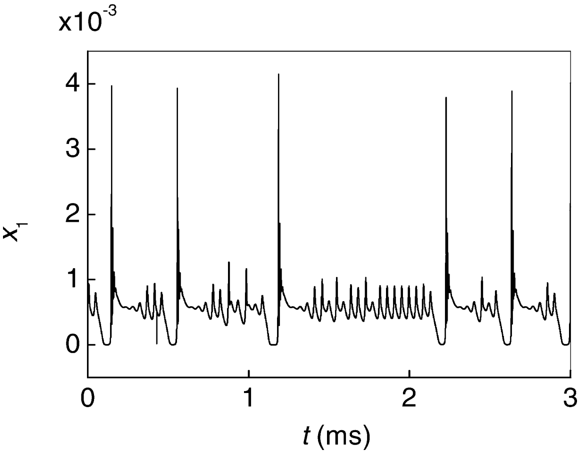


Fig. 8. Time series of chaotic oscillations in a $CO_2$ laser with feedback $R = 160$, $B_0 = 0.10315$.

contains $z = 10$ sublevels. The variable $x_6$ is proportional to the feedback voltage that affects the cavity loss parameter through the relation $k_0(1 + k_1 \sin^2 x_6)$. $\theta$ is the time rescaled to the collision relaxation rate $\gamma_r = 7 \times 10^5\ \mathrm{s}^{-1}$, i.e., $\theta = t\gamma_r$, where $t$ is the real time. The control parameters $B$ and $R$ are proportional to the bias voltage and the gain of the feedback, respectively. The parameters $\Gamma_1$, $\Gamma_2$, $\gamma$, and $\beta$ represent decay rates, $\alpha$ is a saturation factor of the feedback loop, and $P_0$ is the pump parameter. The fixed parameter values are collected in Table 1. They correspond to accurate measurements performed on the experimental system [37].

Eq. (6) contains the delayed variable $\varepsilon x_6(\theta - T_0)$, where $\varepsilon$ is the strength of the delay signal and $T_0$ is the delay time. The laser without delay ($T_0 = 0$ and $\varepsilon = 0$) operates in a chaotic regime which is characterized by the large and small spikes (Fig. 8) corresponding to the alternative large and small loops forming respectively the inward and outward spirals in the phase space in the vicinity of an unstable saddle focus. This fixed point has two real ($\lambda_1$ and $\lambda_2$) and four complex conjugate eigenvalues. The inward spiral motion is related to a nonstationary solution of the equation model characterized by a stable manifold with complex eigenvalues ($-\rho_2 \pm i\omega_2$). For a homoclinic orbit associated with such a saddle focus Shilnikov showed that, if $|\rho_2/\lambda_2| < 1$, then a homoclinic orbit is created and the system represents a chaotic behavior of Shilnikov type [33].

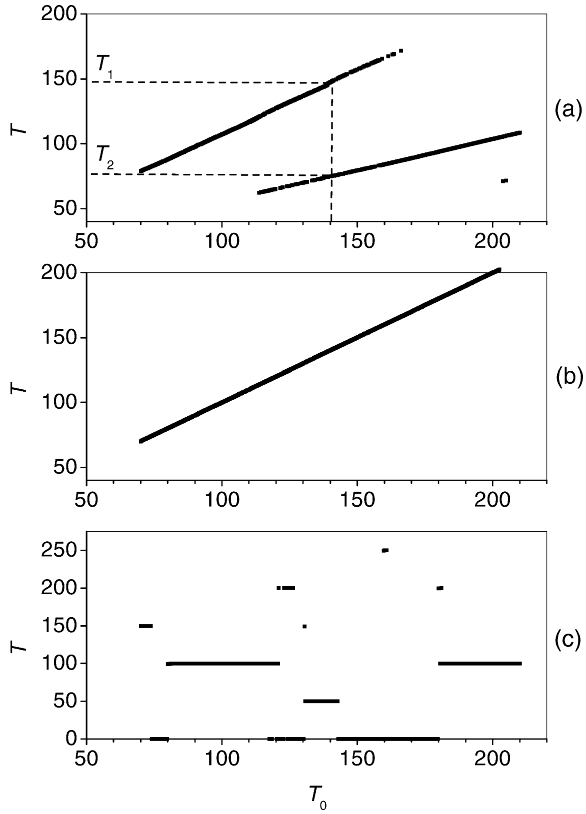


Fig. 9. (a) Coexisting attractors in the presence of time delay in feedback. (b) Locking of the periodic orbit with $T = T_0$ when the variable is modulated with delay-dependent frequency $f = 1/T_0$ and (c) with fixed frequency $f = 1/50$, $\zeta = 0.01$ and $\varepsilon = 0.25$. The horizontal dashed lines show the periodicities of coexisting states, $T_1 = 147.3$ and $T_2 = 74.7$, which appear at the delay with $T_0 = 140$ shown by the vertical dashed line.

Synchronization of Shilnikov chaos by delayed feedback ("delayed self-synchronization") was recently demonstrated by Arecchi et al. [11]. However, at certain delay times this control may induce multiple coexisting attractors, whose period $T$ is a linear function of the delay time $T_0$. For example, two coexisting periodic attractors appear at $\varepsilon = 0.25$ in the range of delay times $110 < T_0 < 160$ (in normalized units of $\gamma_r$), as shown in Fig. 9(a). At fixed $T_0 = 140$ the coexisting attractors have periods $T_1 = 147.3$ and $T_2 = 74.7$ (horizontal dashed lines) that are close to $T_0$ and $T_0/2$. The dependences in Fig. 9 are obtained by calculating the autocorrelation function of the laser intensity and measuring the correlation period [11]. The chaotic regime yields zero autocorrelation function and hence $T = 0$ (Fig. 9(c)). In the next subsection we will show that harmonic modulation to the feedback variable allows locking one of the coexisting periodic orbits.

### 3.2. Modulation of the feedback variable

In the laser with harmonic modulation of the feedback variable Eq. (6) becomes

$$\dot{x}_1 = k_0 x_1 \big\{ x_2 - 1 - k_1 \sin^2\big[\varepsilon x_6(\theta - T_0) + (1-\varepsilon)x_6 + \zeta \sin(2\pi f \theta)\big]\big\}, \qquad (12)$$

where $\zeta$ is the modulation amplitude. The effect of the modulation depends strongly on the modulation frequency. We find that one of the coexisting periodic orbits can be locked when (i) the modulation frequency

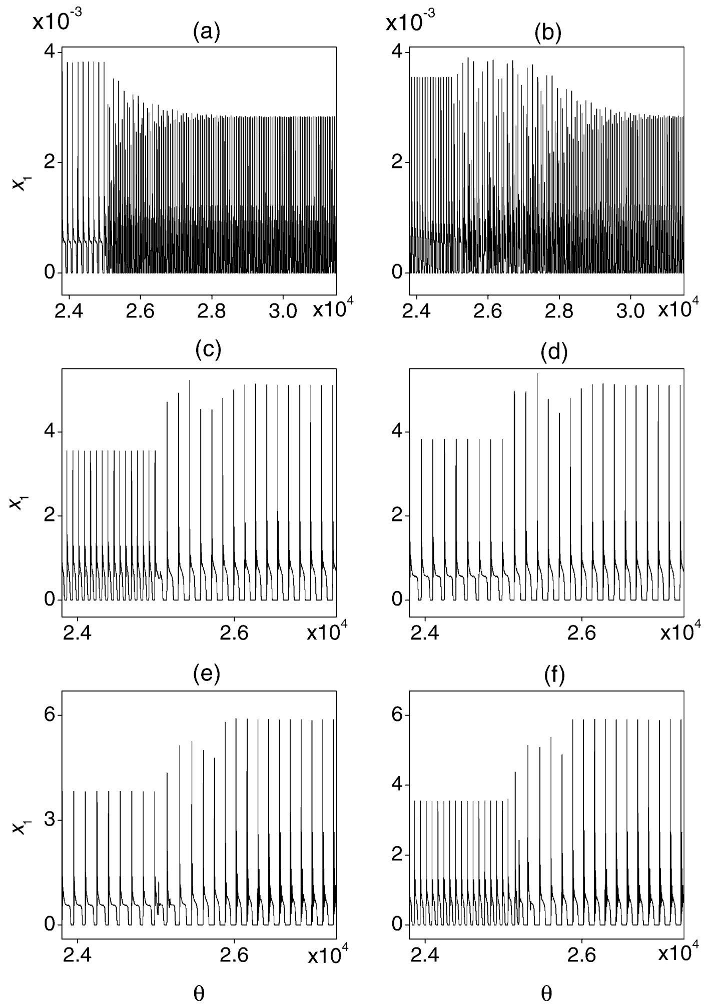


Fig. 10. Time series of the intensity of a $CO_2$ laser when feedback modulation with $\zeta = 0.02$ is applied at time $\theta = 2.5 \times 10^4$ with frequencies (a), (b) $f = 1/50$, (c), (d) $1/T_0$ and (e), (f) $f_r = 7.25 \times 10^{-3}$. The periodicities of the initial states (a), (c), (e) $T_1 = 147.3$ and (b), (d), (f) $T_2 = 74.7$.

is proportional to the reciprocal delay time ($f = n/T_0$, where $n = 1, 2, \ldots$) or (ii) the modulation frequency is fixed ($f = \text{const}$). These cases are illustrated, respectively, in Fig. 9(b) and (c). Note, that $T$, $T_0$, $f$ in the text and figures are normalized to $\gamma_r$. In both cases the modulation amplitude is relatively small ($\zeta = 0.02$) and there are ranges of $T_0$ where only one periodic state exists.

The analysis of these results allows us to reveal the following main features of this kind of the control.

(i) Time dependent modulation of the delay variable locks the state whose period is equal to $T_0$, i.e., $T = T_0$ (Fig. 9(b)).
(ii) Fixed frequency modulation locks a periodic orbit multiple of $T_0$, i.e., $T = (i/j)T_0$ $(i, j = 1, 2, \ldots)$ (Fig. 9(c)).

Fig. 10 demonstrates the effect of the control modulation with the time series. The delay time is fixed at $T_0 = 140$ and the control is switched on at time $\theta = 2.5 \times 10^4$. In the left-hand column the control is applied to the orbit with period $T_1 = 147.3$ and in the right-hand column to the orbit with $T_2 = 74.7$. One can see that after transients both attractors undergo crisis that leads to their destruction and the trajectory

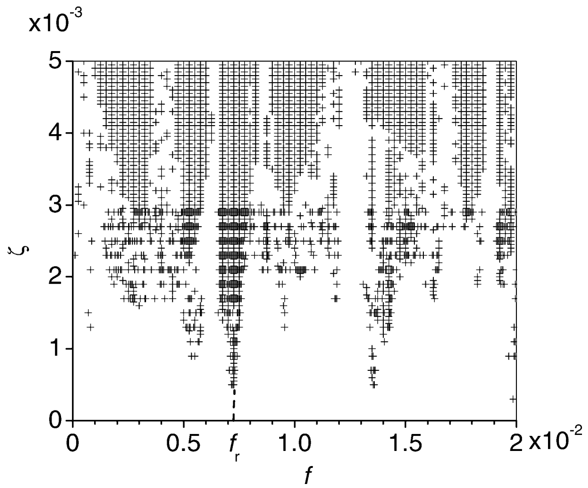


Fig. 11. Stability boundary of the period-74.7 attractor in the $CO_2$ laser with periodic modulation of feedback variable. The crosses indicate the range where the period 74.7 does not exist. For the modulation parameters below the boundary two attractors with periods 74.7 and 147.3 coexist.

is attracted to the attractor whose period is equal to the period of the modulation.

In Fig. 11 we show the stability boundary (annihilation curve) for the attractor with period $T_2 = 74.7$ in the space of modulation parameters $\zeta$ and $f$. The crosses indicate the region in the parameter space where the attractor $T_2 = 74.7$ does not exist. This region forms tongues whose minima are located at the relaxation oscillation frequency of the attractor, $f_r \approx 7.25 \times 10^{-3}$ and at the frequencies equal to approximately second and third harmonics of $f_r$. For the parameters inside the tongues (above the annihilation curve), attractor $T_2 = 74.7$ is destroyed and only attractor $T_1 = 147.3$ remains. Thus, the modulation of the feedback variable makes the system monostable.

## 4. Conclusions

In this Letter we have introduced a method for stabilization of a particular periodic orbit in a chaotic system with feedback. First, we applied a time delay in feedback which allowed us to stabilize several periodic orbits embedded into the chaotic attractor and then we added a harmonic modulation to the feedback variable to lock one of the periodic states and by such a way to make the system monostable. The method has been demonstrated with a simple chaotic map (logistic map) and a laser model described a $CO_2$ laser operating in a chaotic regime of Shilnikov type.

The control of dynamical systems with multistability in order to destroy some of coexisting attractors while preserving the others is extremely important, especially for laser physics, where the multistability plays a key role in limitation of the performance characteristics. Recently, the control of multistability in nonautonomous systems with coexisting attractors has been realized in semiconductor [38] and fiber lasers [39] with modulated pump parameter. This Letter has demonstrated that multistability can be also efficiently controlled in autonomous systems by modulating feedback variable instead of modulation of a system parameter. We believe that the approach described in this Letter may have a general application to control multistability in nonlinear dynamical systems with feedback.

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