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Critical diffusion behaviour of a climbing-sine map near intermittency threshold

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The diffusion in a climbing-sine map near intermittency threshold is considered. A simple description is proposed for particles dispersion at small and moderate times when the model reveals a crossover from anomalous to normal diffusion and shows oscillations in the time dependence of low moments of the probability distribution function. For larger times when flights of particles decorrelate, a different approach is employed that allows one also to take into account the external noise. A universal scaling function is found describing the critical behaviour of the diffusion constant under external noise near the intermittency threshold. The results of numerical simulations are in agreement with the analytical formulas.

1. Introduction

It is well known that particles in regular fields may move chaotically. Sometimes this motion is diffusive, i.e. the particle moves infinitely far from its initial position with time [1–9]. The mean-square particle displacement may grow asymptotically either as a linear (normal diffusion) or nonlinear function in time (anomalous diffusion). The most complete description of the diffusion process is provided by the probability distribution $P(r,t)$ to find the particle at time t at a distance r from the initial position. However, the function $P(r,t)$ is not always convenient for the description of diffusion in experiment or in numerical simulations since it may be rather complicated or even fractal. The approach based on the calculation of scaling diffusion characteristics [1] appears to be a better tool. Computing the moments $\langle r^q \rangle$ for different q one can find a unique function $\gamma(q)$ that is determined by the asymptotic expression

$$\langle r^q \rangle \propto t^{q\gamma(q)}$$

as $t \to \infty$. Examples for using the generalized diffusion exponent $\gamma(q)$ in the description of the diffusion of passive particles in hydrodynamic wave fields are given in [1,2].

Diffusive motion of particles is also possible in simple one-dimensional maps. The so-called climbing-sine map

$$x_{n+1} = x_n + E \sin 2\pi x_n \tag{1}$$

was considered in [3,4] where it was shown that the chaotic sequence x_n leaves the interval $[0,1]$ when $E > E_0 = 1.73264...$ and $x_0 \in [0,1]$ and as $n \to \infty$, $\langle x_n^2 \rangle \propto 2Dn$. For $0 < E - E_0 \ll E_0$, a scaling of the diffusion constant

$$D = 0.371(E - E_0)^{1/2} \tag{2}$$

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was found [3,4]. The effect of additive noise on the onset of diffusion was also considered in [4] and a universal scaling function describing this transition and depending only on the combination $(E - E_0)/\sigma$ (where $\sigma \ll 1$ is the standard deviation of Gaussian noise) was found.

It was pointed out in [3] that for $E \to E_1 = 1$ there appears in the climbing-sine map a region of intermittency that leads to divergence in the diffusion constant

$$D \cong \frac{1}{2\sqrt{2\varepsilon}}, \tag{3}$$

where $\varepsilon = 1 - E$. A reason for this divergency is the presence of very long laminar "flights" of the length $T_0 \propto \mathcal{O}(\varepsilon^{-1/2})$ interrupted by short chaotic bursts.

A more detailed study of the intermittent diffusion was performed by Geisel and Nierwetberg in 1984 [5,6]. They calculated various statistical characteristics like velocity correlation function, power spectrum, mean-square displacement and found a crossover behavior when below a crossover line the mean-square displacement grows like n^2 and normal diffusion begins larter on. The procedure of statistical averaging proposed in [5,6] implicitly uses the assumption that the ergodicity is set once at $n = 0$ and the number of particles which are in the stage of flight does not depend on n. This assumption is reasonable if one averages over various pieces of one very long particle's trajectory. If however one is averaging over the equiprobable initial locations of the particle inside the interval [0,1], this assumption does not work initially and ergodicity establishes only after a long time $\mathcal{O}(\varepsilon^{-1})$ which for small ε is much longer than the length of the flight T_0. That is why one needs different models for the initial stage of intermittent diffusion and for its final stage.

The present paper is concerned with the latter formulation of the problem, when the averaging is carried out over initial locations. Results of numerical calculations (see section 2) show that as in [5,6], there is a crossover in the dependence of $\langle x_n \rangle$ on n such that for not too large n $(\langle x_n^q \rangle)^{1/q} \propto n$, while for $n \to \infty$, $(\langle x_n^q \rangle)^{1/q} \propto n^{1/2}$. A characteristic feature in the behaviour of the moments $\langle x_n^q \rangle$ for small q is now the presence of quasi-regular oscillations when n is not too large. In section 2 we propose a simple statistical approach that allows for the description of both these oscillations and the crossover.

After a long transient period ergodicity eventually establishes and an approach used in [5,6] starts to work for our problem. Nevertheless, we employ for this late stage the technique used in [4], which also permits including additive noise (section 3). In this case, there is no divergence in the diffusion constant as $E \to 1$, instead it appears as a universal function of the parameters ε and σ.

2. Statistical model of particle dispersion in a "correlated" regime

There are three types of particle motions described by the climbing-sine map (1) depending on the amplitude E: localized (regular or chaotic) motion, motion with constant velocity (running solutions):

$$x_{n+1} = x_n \pm m, \quad m = 1, 2, \ldots \tag{4}$$

and diffusive random walks along the x-axis. In the neighbourhood of the integral values of the amplitude $E \neq 0$, a tangential bifurcation occurs with a transition from diffusive to running motion. In terms of nonlinear dynamics this is a transition through intermittency of type I [10]. Near the first transition from diffusion to the running solutions at $E \leq 1$, the map (1) taken modulo 1 has the form shown in fig. 1. Intermittency appears near the points $x_\pm = \frac{1}{4}, \frac{3}{4}$ where the map (1) (mod 1) can be represented approximately in the form

$$\chi_{n+1} = \chi_n \pm 2\pi\chi_n^2 \pm \varepsilon, \tag{5}$$

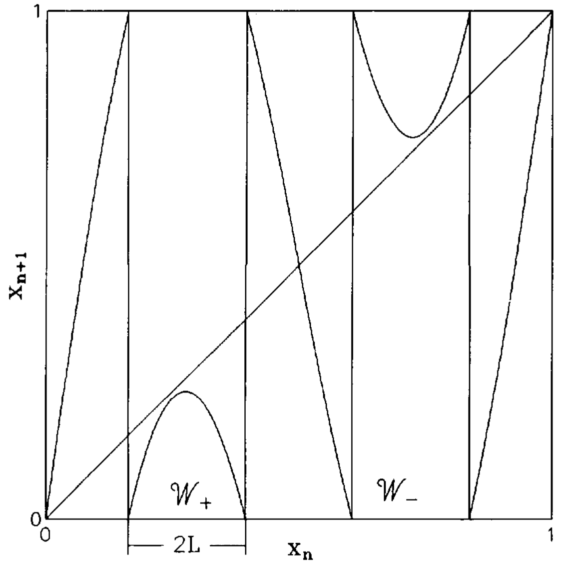

Fig. 1. Climbing-sine map (1) taken *modulo* 1 for $E = 0.99$ ($\varepsilon = 10^{-2}$). W_+ and W_- are basins of laminar motion in + and − directions, respectively.

where $\chi_n = x_n - x_\pm$. When $\varepsilon \ll 1$, the image point is for a while in the vicinity of the points $x_\pm$ corresponding to constant-velocity motion along the positive or negative x-axis. The maximal length of these flights is determined in the resonant approximation [3] from the differential equation

$$\frac{d\chi}{dt} = 2\pi^2\chi^2 + \varepsilon \tag{6}$$

and is equal to $X_0 = (2\varepsilon)^{-1/2}$. After escaping the neighbourhood of the intermittency point, the particle performs a few chaotic oscillations before arriving again in the vicinity of one of the intermittency point $x_\pm$ to start a new flight. It can readily be shown (see,e.g., [3,11]) that the distribution function for the lengths of the flights has approximately the form

$$P(X) \cong \frac{1}{4\pi^2 L}\{1 + \cot^2[\pi(2\varepsilon)^{1/2}X]\}, \tag{7}$$

where L is the halfwidth of the regions $W_\pm$ of laminar flights within mapping (1) (see fig. 1). In particular, it follows from (7) that the average length of the flights is exactly half the maximal length, i.e. $X_{\text{mean}} = \frac{1}{2}(2\varepsilon)^{-1/2}$.

It is natural for the calculation of the diffusion characteristics of the climbing-sine map (1) to consider an ensemble of particles with a uniform initial distribution on the interval [0,1]. We have already noted that when $\varepsilon \ll 1$, all the particles move quickly to the neighbourood of the intermittency points $x_\pm$, i.e. they will soon find themselves in the "running" regime.

We will make the following further approximation: we replace the distribution function (7), having for $\varepsilon \ll 1$ two pronounced maxima, by a sum of two δ-functions:

$$P(X) = \tfrac{1}{2}[\delta(X) + \delta(X - X_0)] \tag{8}$$

In this approximation, at $t = \mathcal{O}(1)$ one half of the particles starts flights X_0 long, while the other half immediately escapes to the region of chaotic walks. This second group of particles is quite soon (in the time $\mathcal{O}(1)$) reinjected into the region of intermittency, half of them starting a long flight and the other half being reinjected again, and so on. Thus, after the time $\mathcal{O}(1)$ nearly all the particles will be at the initial stage of the flight but half of them will move to the positive while the other half to the negative direction of the x-axis. After the time $T_0 (= X_0)$, all the particles will escape from the intermittency regions to the region of stochasticity and after another the time $\mathcal{O}(1)$ they will be reinjected into the intermittency region and will start new flights. It is easy to see that within this rather rough approximation, the problem of the diffusion in a climbing-sine map reduces to the problem of a particle that in equal time intervals jumps with equal jump lengths in equiprobable directions. (This problem is well known from probability theory.) It corresponds to the Bernoulli scheme and the distribution function of the particles at time $t = nT_0$ is described by the binomial distribution

$$P(x,nT_0) = \frac{1}{2^n}\sum_{l=0}^{n} C_n^l \delta(x-(n-2l)T_0). \tag{9}$$

Substituting the sum

$$\tfrac{1}{2}[\delta(x-kT_0-\tau)+\delta(x-kT_0+\tau)],$$

instead of $\delta(x-kT_0)$ in (9), we obtain the distribution function of the particles at arbitrary time t:

$$P(x,t) = \frac{1}{2^{n+1}}\sum_{l=0}^{n} C_n^l[\delta(x-(n-2l)T_0-\tau) + \delta(x-(n-2l)T_0+\tau)], \tag{10}$$

where $\tau = t(\text{mod } T_0)$ and $n = (t-\tau)/T_0$. Knowing the probability distribution (10) one can readily calculate its qth moment

$$\langle|x|^q\rangle = \frac{1}{2^{n+1}}\sum_{l=0}^{n} C_n^l[|(n-2l)T_0-\tau|^q + |(n-2l)T_0+\tau|^q] \tag{11}$$

It follows from (11) that for $t < T_0$ we have

$$M_q = [\langle|x|^q\rangle^{1/q} = t,$$

which corresponds to anomalous diffusion with $\gamma(q) = 2$. For $t > T_0$, M_q increases monotonically with t when $q > 2$ and oscillates when $q < 2$ with the local minima attained at $t = 2lT_0$. For large t the envelope of the function $M_q(t)$ asymptotically approaches $(2Dt)^{1/2}$ with

$$D = \tfrac{1}{2}T_0 = \tfrac{1}{2}(2\varepsilon)^{-1/2}.$$

(It is of no importance whether the envelope passes through the minima or the maxima of the function $M_q(t)$). Thus, the model shows a crossover between anomalous diffusion with $\gamma(q) \equiv 2$ when $t \le T$ and normal diffusion when $t \gg T_0$. In the latter case, the diffusion coefficient diverges as $\varepsilon^{-1/2}$ when $\varepsilon \to 0$ and coincides, up to a numerical coefficient, with the expression (3) that was obtained in [3] from other considerations.

Of course, the model proposed is rather rough, but numerical simulation of particle diffusion in a climbing-sine map show good agreement between theory and the data of our numerical experiments. The calculations were performed for 1000 particles distributed uniformly on the [0,1] interval at time $t = 0$. Plots of the functions $M_{1/2}(t)$ and $M_2(t)$ obtained in the numerical experiments with $\varepsilon = 10^{-6}$ and the theoretical curves obtained from (11) are presented in figs. 2a, 2b.

3. Diffusivity in a climbing-sine map with external noise

As was pointed out in the introduction, the model presented in the previous section is not valid for sufficiently large t. The reason is that a small spreading in the duration of flights and of the random walks between the flights leads in the time $\mathcal{O}(\varepsilon^{-1})$ to a complete decorrelation of the particles such that the number of particles, which perform a flight simultaneously, no longer depends on time. In this case, however, a different approach may be proposed which can also include additive noise effects on the diffusion characteristics of the particles motion in the climbing-sine map near intermittency threshold.

Thus, consider a generalized climbing-sine map

$$x_{n+1} = x_n + E\sin 2\pi x_n + \xi, \tag{12}$$

where ξ is a random variable with Gaussian distribution $\nu(\xi)$ and a standard deviation $\sigma \ll 1$. Following [4] we will introduce the distribution function $\rho_t(x)$ which obeys the equation

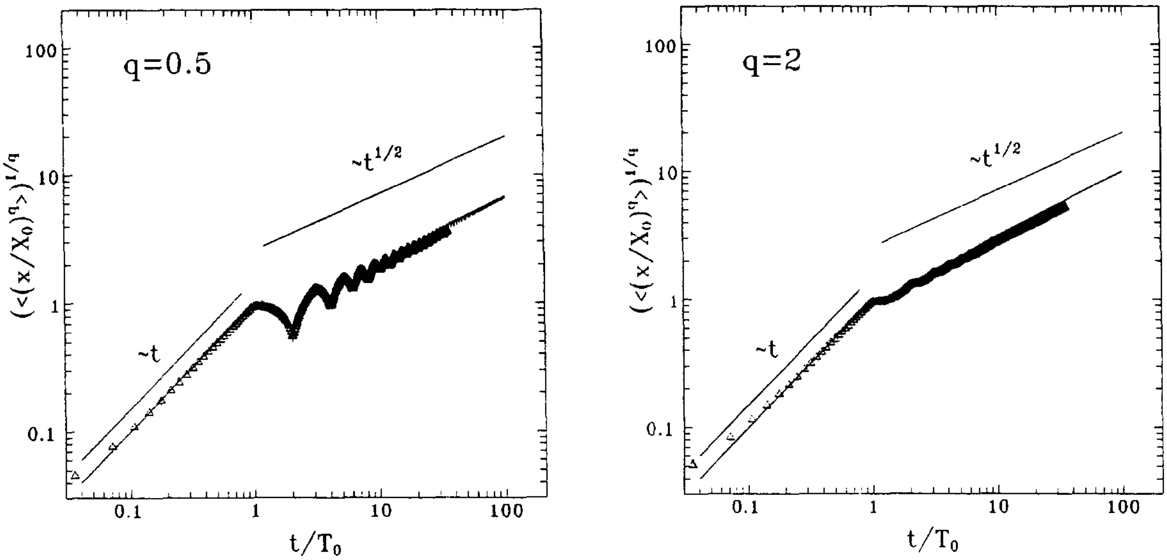

Fig. 2. Time dependence of the moments $M_{1/2}(t)$ (a) and $M_2(t)$ (b) of the probability distribution function for particles in a "correlated" regime: solid lines show formula (11); triangles (Δ) correspond to the results a numerical simulation for $\varepsilon = 10^{-6}$.

$$\rho_{t+1}(x) = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \rho_t(x)\,\nu(\xi) \times \delta(x - f(x) - \xi)\,\mathrm{d}\xi, \tag{13}$$

(in our case $f(x) \equiv E \sin 2\pi x$). Integrating (13) with respect to x from l to $l+1$, we obtain the probability for a particle to be in the lth unit interval (lth cell) of the x-axis at time $t+1$ [4]:

$$p_{t+1}(l) \equiv \int_{l}^{l+1} \rho_t(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{d}x \int_{g(x}^{s(x)} \rho_t(x)\nu(\xi)\,\mathrm{d}\xi, \tag{14}$$

where $g(x) = [l - f(x)]/\sigma$ and $s(x) = [l + 1 - f(x)]/\sigma$. Let us divide the probability distributions $\rho(x)$ and $p_t(l)$ into two parts

$$\rho_t(x) = \rho_t^+(x) + \rho_t^-(x),$$
$$p_t(l) = p_t^+(l) + p_t^-(l), \tag{15}$$

where + and − correspond to the distribution of the particles that arrived at the cell l from left or right, respectively. With (14) taken into account, it is easy to show that p^+ and p^- obey the equations

$$p_{t+1}^+(l) = \int_{l+1}^{\infty} \mathrm{d}x \int_{g(x)}^{s(x)} \rho_t(x)\nu(\xi)\,\mathrm{d}\xi + \int_{l}^{l+1} \mathrm{d}x \int_{g(x)}^{s(x)} \rho_t^+(x)\nu(\xi)\,\mathrm{d}\xi, \tag{16}$$

$$p_{t+1}^-(l) = \int_{-\infty}^{l} \mathrm{d}x \int_{g(x)}^{s(x)} \rho_t(x)\nu(\xi)\,\mathrm{d}\xi + \int_{l}^{l+1} \mathrm{d}x \int_{g(x)}^{s(x)} \rho_t^-(x)\nu(\xi)\,\mathrm{d}\xi. \tag{17}$$

When $\sigma \ll 1$, the integration with respect to ξ in (16) and (17) may be performed explicitly up to terms of the order of $\sigma \exp(-\frac{1}{2}\sigma^2)$. The resulting equations for $p_t^\pm(l)$ (cf. [4]) have the following form:

$$p_{t+1}^+(l) - p_t^+(l) = -\frac{1}{2}\int_l^{l+1} \rho_t^+(x)\{\mathrm{erfc}[2^{-1/2}s(x)] + \mathrm{erfc}[-2^{-1/2}g(x)]\}\,dx + \frac{1}{2}\int_{l+1}^{l+2} \rho_t^+(x)\{\mathrm{erfc}[2^{-1/2}g(x)] - \mathrm{erfc}[2^{-1/2}s(x)]\}\,dx + \frac{1}{2}\int_{l+1}^{l+2} \rho_t^-(x)\{\mathrm{erfc}[2^{-1/2}g(x)] - \mathrm{erfc}[2^{-1/2}s(x)]\}\,dx, \quad (18)$$

$$p_{t+1}^-(l) - p_t^-(l) = -\frac{1}{2}\int_l^{l+1} \rho_t^-(x)\{\mathrm{erfc}[2^{-1/2}s(x)] + \mathrm{erfc}[-2^{-1/2}g(x)]\}\,dx + \frac{1}{2}\int_{l+1}^{l+2} \rho_t^-(x)\{\mathrm{erfc}[2^{-1/2}g(x)] - \mathrm{erfc}[2^{-1/2}s(x)]\}\,dx + \frac{1}{2}\int_{l+1}^{l+2} \rho_t^+(x)\{\mathrm{erfc}[2^{-1/2}g(x)] - \mathrm{erfc}[2^{-1/2}s(x)]\}\,dx, \quad (19)$$

where $\mathrm{erfc}(x)$ is a conjugate error function. For large times the ratios $\rho_t^\pm(x)/p_t^\pm(l)$ cease to depend on t and l and converge to the invariant probability distributions $\rho^\pm(x)$. Then, employing also the property of translational invariance of the function $f(x)$ (see [4]), eqs. (18)–(19) can be transformed to

$$p_{t+1}^+(l) - p_t^+(l) = \Gamma_f[p_t^+(l+1) - p_t^+(l)] + \Gamma_b[p_t^-(l+1) - p_t^+(l)], \quad (20)$$

$$p_{t+1}^-(l) - p_t^-(l) = \Gamma_f[p_t^-(l-1) - p_t^-(l)] + \Gamma_b[p_t^+(l-1) - p_t^-(l)], \quad (21)$$

where

$$\Gamma_f = \frac{1}{2}\int_0^1 \rho^+(x)\,\mathrm{erfc}[2^{-1/2}f(x)/\sigma]\,dx, \quad (22)$$

$$\Gamma_b = \frac{1}{2}\int_0^1 \rho^-(x)\,\mathrm{erfc}[2^{-1/2}f(x)/\sigma]\,dx, \quad (23)$$

Relations (20), (21) constitute a system of discrete master equations. A continuous analog of them was employed, in particular, in [12] for the solution of a particle performing a walk on a one-dimensional lattice with the probability for retaining the direction of motion in the next time step equal to Γ_f and the probability for changing it equal to Γ_b.

It follows from the results of [10] that for large times the particle's mean-square displacement from the initial state grows linearly with time:

$$\langle x^2\rangle \propto 2Dt$$

with the diffusion coefficient

$$D = \frac{\Gamma_f}{\Gamma_b}(\Gamma_f + \Gamma_b)a^2, \quad (24)$$

where a is the mesh of the lattice. In our case $a = 1$ and $\Gamma_f + \Gamma_b = 1$, so that $D = \Gamma_f/\Gamma_b$. Explicit calculation of D needs knowledge of the invariant probability distributions $\rho^\pm(x)$.

The invariant probability distributions $\rho^\pm(x)$ near the intermittency points $x_\pm$ can be found from the stationary Fokker–Planck equation

$$\frac{d}{dx}[\phi(x)\rho^\pm] - \sigma^2\frac{d^2\rho^\pm}{dx^2} = r(x), \quad (25)$$

where $\phi(x) = f(x)(\mathrm{mod}\ 1)$, and $r(x)$ denotes the reinjection rate which is equal to the particle flux through the intermittency region. The invariant probability distributions have pronounced maxima at $x_\pm = \frac{3}{4}, \frac{1}{4}$ near the intermittency threshold ($\varepsilon \ll 1$) and for $\sigma \ll 1$. Substituting the variables $\chi_\pm = x - x_\pm$ and expanding the function $\phi(x)$ in the vicinity of these points, eq. (25) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\chi_\pm}[\pm(2\pi^2\chi_\pm^2 + \varepsilon)\rho^\pm] - \sigma^2\frac{\mathrm{d}^2\rho^\pm}{\mathrm{d}\chi_\pm^2} = r. \tag{26}$$

At $\varepsilon \ll 1$ and $\sigma \ll 1$ the gradient of the particle flux r can be neglected in the region of localization of the function $\rho^\pm(x)$. Then eq. (26) has the following solution:

$$\tilde{\rho}^\pm(\tilde{\chi}_\pm) = \pm\frac{C_\pm}{\sigma^{2/3}}\int_0^\infty \exp[-\zeta(2\pi^2\tilde{\chi}_\pm^2 + \delta) + 2\pi^2\zeta^2\tilde{\chi}_\pm^2 - \tfrac{2}{3}\pi^2\zeta^3]\,\mathrm{d}\zeta, \tag{27}$$

where $\delta = \varepsilon/\sigma^{4/3}$ and the scaled variables $\tilde{\rho}^\pm \equiv \sigma^{2/3}\rho^\pm$ and $\tilde{\chi}_\pm \equiv \sigma^{-2/3}\chi_\pm$ have been used. The integration constant $C\pm$ is calculated from the normalization condition

$$\int_{-\infty}^{\infty} \rho^\pm(\chi_\pm)\,\mathrm{d}\chi_\pm = 1, \tag{28}$$

which yields

$$C_\pm = \pm\sigma^{2/3}\sqrt{2\pi} \times\left(\int_0^\infty \frac{\mathrm{d}\zeta}{\sqrt{\zeta}}\exp[-\delta\zeta - \tfrac{1}{6}\pi^2\zeta^3]\right)^{-1}. \tag{29}$$

Invariant probability distributions $\tilde{\rho}^\pm(\tilde{\chi}_\pm)$ at different δ's are presented in fig. 3. If the total number of particles is N, then $\frac{1}{2}C_\pm N$ denotes the particle flux near the intermittency points $x_\pm$ (for $\chi_\pm = 0$). It is easily seen that for $\varepsilon \ll 1$ and $\sigma \ll 1$ $C_\pm$ is related to $r(x)$ within an accuracy of $\mathcal{O}(\exp(-\frac{1}{2}\sigma^2))$ by the formula

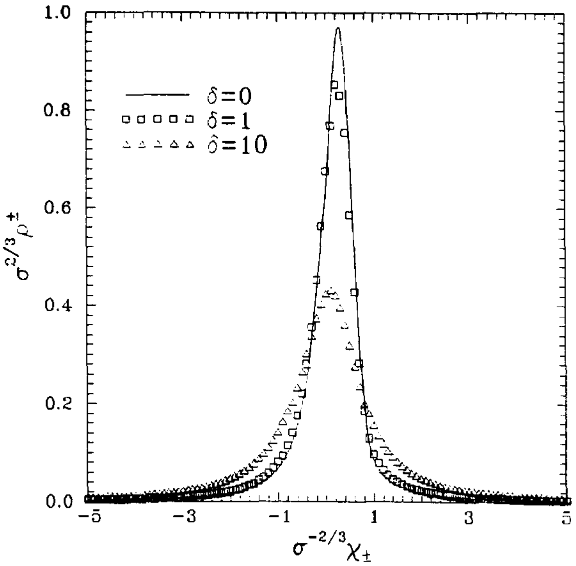

Fig. 3. Invariant probability distribution $\rho(x)$ in the vivinity of the intermittency points $x_\pm$ for three values of the parameter $\delta = \varepsilon/\sigma^{4/3}$.

$$C_\pm = \int_{x'_\pm}^{x_\pm} r(x)\,\mathrm{d}x, \tag{30}$$

where $(x'_\pm, x_\pm)$ are the basins of attraction of the intermittency points $x_\pm$, and $x'_\pm$ as well as $x_\pm$ satisfy the equations

$$x + \sin 2\pi x = \tfrac{5}{4}, -\tfrac{1}{4}. \tag{31}$$

It follows from (30) that for small ε, σ the reinjection rate $r(x)$ scales as $C_\pm$, so that one can introduce an universal function $R(x)$ through

$$R(x) = C_\pm^{-1} r(x) \tag{32}$$

(see fig. 4).

We can now determine Γ_f and Γ_b. First of all note that for $\sigma \ll 1$ the function $\mathrm{erfc}[2^{-1/2}f(x)/\sigma]$ takes approximately only two values: it is equal to 2 inside the region W_+ and zero elsewhere. Actually, this reduces expressions (22) and (23) to the form

$$\Gamma_\mathrm{f} = \int_{W_+} \rho^+(x)\,\mathrm{d}x,$$

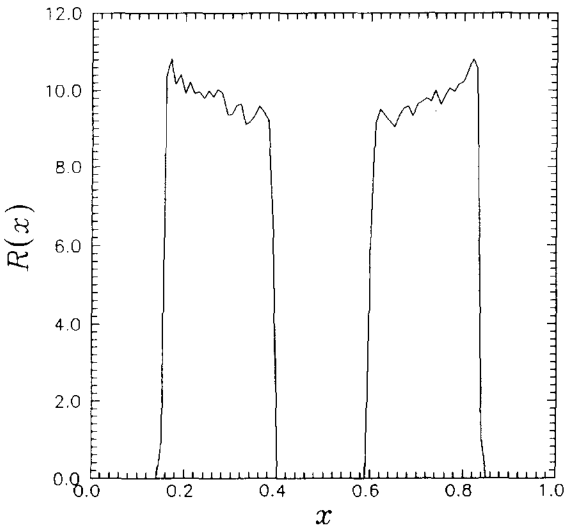

Fig. 4. Scaled rate $R(x)$ of reinjection into the basins $W_\pm$ for the climbing-sine map at $\varepsilon = 1 - E, \sigma \ll 1$.

$$\Gamma_b = \int_{W_+} \rho^-(x)\,dx. \qquad (33)$$

Taking into account the sharpness of the invariant distribution $\rho^+(x)$ (see (27)) we obtain $\Gamma_f \cong 1$ with the exponential accuracy. From (33) it follows that Γ_b is the ratio of the number of particles ΔN that have escaped from basin W of cell $l-1$ to basin W of the cell l in one iteration to the total number $N/2$ of particles that have arrived to cell l from the cell $l-1$. Apparently, ΔN equals half of the number of particles that have fallen within W_+ of cell l in one iteration (the other half are the particles that have fallen within W_+ from cell $l+1$. Thus,

$$\Gamma_b = 2\int_{W_\pm} r(x)\,dx = 2|C_\pm| \int_{W_\pm} R(x)\,dx$$

and the final expression for the diffusion coefficient near the intermittency threshold has the form

$$D = A\sigma^{-2/3} d(\varepsilon/\sigma^{4/3}) \qquad (34)$$

with

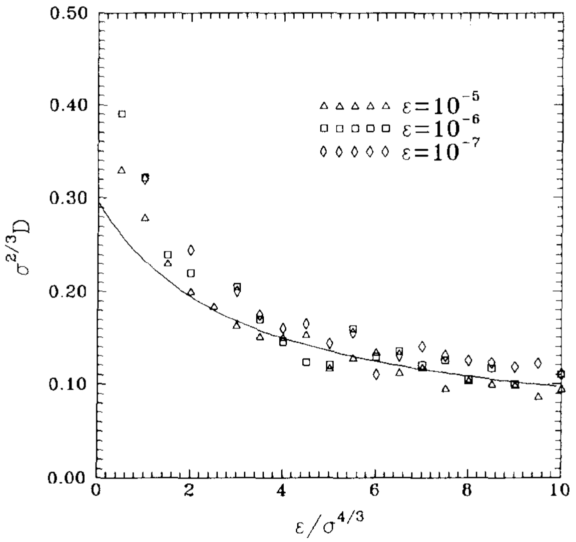

Fig. 5. Universal scaling function (36) and data of computer simulation for the generalized climbing-sine map (12) for different values of the parameter ε and noise strength σ.

$$d(\delta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{d\zeta}{\sqrt{\zeta}} \exp[-\delta\zeta - \tfrac{1}{6}\pi^2\zeta^3], \qquad (35)$$

$$A = \left(\int_{W_+} R(x)\,dx\right)^{-1} \cong 0.43. \qquad (36)$$

These formulas show that $\sigma^{2/3}D$ is described by a universal scaling function that depends only on the parameter $\delta = \varepsilon/\sigma^{4/3}$, as is typical in the theory of phase transitions. For small δ, $D = A\sigma^{-2/3}d(0) \cong 0.29\sigma^{-2/3}$. When $\delta \gg 1$, $d(\delta) \cong (2\pi d)^{-1/2}$ and D shows the same asymptotic behavior as in (3) but with a slightly different numerical constant. A plot of the function $d(\delta)$ in (36) for $0 < \delta < 10$ is shown in fig. 5 together with data of a numerical simulation of the climbing-sine map with Gaussian noise (12). The numerical calculations were performed for different values of σ and of the parameter δ in order to check the scaling law (34), (36) in the diffusion behavior of the climbing-sine map with noise (12). One can see from fig. 5 that, in accordance with (33), all the points in the $(\sigma^{2/3}D, \varepsilon/\sigma^{4/3})$ plot are near one curve that agrees well with the universal scaling function (36) at large enough values of δ but is somewhat underestimated by (36) at $\delta < 1$.

4. Conclusions

In this paper we have considered particle dispersion ruled by the climbing-sine map (1) near the intermittency threshold $\varepsilon = 1 - E \ll 1$. The numerical simulations show that for not too large times $t < T_0 = (2\varepsilon)^{-1/2}$ the mean-square particle displacement grows as t^2 while for larger t, $\langle x^2 \rangle \propto t$. The same behavior characterises all other moments $M_q = (\langle x^q \rangle)^{1/q}$. In the crossover region there are a number of quasi-regular oscillations with period $2T$. A simple statistical model proposed in the section 2 allows to describe the crossover as well as the oscillations employing an approach, in which all particles perform flights of equal length T starting simultaneously in any of two directions. This last assumption fails for large time $t = \mathcal{O}(\varepsilon^{-1})$ due to decorrelation of particles.

For these large times regime the alternative approach is developed in section 3 which assumes a total decorrelation of particles and the formation of an invariant probability distribution $\rho(x)$. This model also takes into account external Gaussian noise. As it was pointed out in 4, for small ε there is the critical exponent $-\frac{1}{2}$ and therefore the divergency in the ε-dependence of the diffusion constant: $D \propto \varepsilon^{-1/2}$. With external noise taken into account the diffusion constant does not diverge for $\varepsilon \to 0$. Instead, we have obtained the universal scaling function $\sigma^{2/3} D = d(\varepsilon/\sigma^{4/3})$, where σ is the standard deviation of the noise. In the limit $\delta = \varepsilon/\sigma^{4/3} \to \infty$ this function recovers the critical exponent of D: $D \propto \varepsilon^{-1/2}$. In the opposite limit, $\delta \to 0$, D shows crotical behavior in the intensity σ of noise: $d \propto \sigma^{-2/3}$. The results of our numerical simulations for the climbing-sine map with noise (12) are well in agreement with the models proposed.

It is worth mentioning that statistic properties of intermittency in a logistic map with additional Gaussian noise was studied in [13,14]. It turns out that our universal function which scales the diffusion constant in a climbing-sine map up to notations and numerical constant coincides with the function which describe the average length of laminar segments in the logistic map obtained in those papers by different way. The very fact that different statistical characteristics of different maps are essentially scaled by the same universal function reflects the generic property of the scale invariance near the threshold of intermittency.

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