Dynamics in a Bistable-Element-Network with Delayed Coupling and Local Noise

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Abstract. The dynamics of an ensemble of bistable elements under the influence of noise and with global time-delayed coupling is studied numerically by using a Langevin description and analytically by using 1) a Gaussian approximation and 2) a dichotomic model. We find that for a strong enough positive feedback the system undergoes a phase transition and adopts a non-zero stationary mean field. A variety of coexisting oscillatory mean field states are found for positive and negative couplings. The magnitude of the oscillatory states is maximal for a certain noise temperature, i.e., the system demonstrates the phenomenon of coherence resonance. While away form the transition points the system dynamics is well described by the Gaussian approximation, near the bifurcations it is more adequately described by the dichotomic model.

INTRODUCTION

Stochastic rate processes in bi- or multi-stable systems lead to many interesting phenomena observed in various scientific areas ranging from physics to social science, and have thus been studied for a long time.

Here we consider a network of stochastically driven bistable elements whose distinct feature is a time-delayed coupling. The time delays are considered as uniform and the network elements are assumed to be highly interconnected, so that the connectivity can be approximated by a global all to all coupling.

The dynamics of the network is numerically explored by using a Langevin model and analytically by using a Gaussian approximation and a dichotomous model, derived from the corresponding Fokker-Planck equations and Master equation, respectively.

LANGEVIN MODEL

The Langevin model consists of N equations, each describing the overdamped motion of a particle in a bistable potential in the presence of noise and global coupling to a time-delayed mean field $X(t-\tau) = N^{-1}\sum_{i=1}^{N} x_i(t-\tau)$,

$$\dot{x}_i(t) = x_i(t) - x_i(t)^3 + \varepsilon X(t-\tau) + \sqrt{2D}\xi(t), \qquad (1)$$

where τ is the time delay, ε is the coupling strength of the feedback and D denotes the variance of the Gaussian fluctuations $\xi(t)$.

GAUSSIAN APPROXIMATION

In order to theoretically study the dynamical properties of a globally coupled set of noisy bistable elements (with no time delay), Desai and Zwanzig [1] derived a hierarchy of equations for the cumulant moments of the distribution function from the multi-dimensional Fokker-Planck equation for the joint probability distribution for all elements. For large noise intensities, when the statistics of individual elements are approximately Gaussian, this hierarchy can be truncated. Applying this approach to our system yields the following set of equations for the mean field X and the variance $M = N^{-1}\sum(x_i - X)^2$,

$$\begin{aligned} \dot{X} &= X - X^3 - 3XM + \varepsilon X(t-\tau), \\ \frac{1}{2}\dot{M} &= M - 3X^2M - 3M^2 + D. \end{aligned} \qquad (2)$$

DICHOTOMOUS MODEL

To study the dynamics of a single bistable element with time-delayed feedback Tsimring and Pikovsky [2] used a dichotomous approximation. In the dichotomous approximation intra-well fluctuations of x_i are neglected, so that a bistable element can be replaced by a discrete two-state system. Applying this complementary approach to our system, allows us to express the mean field dynamics in terms of the hopping rates $p_{12,21}$ which denote the probability of a bistable element to change its state form -1 to $+1$ and vice versa, respectively. The equation for the

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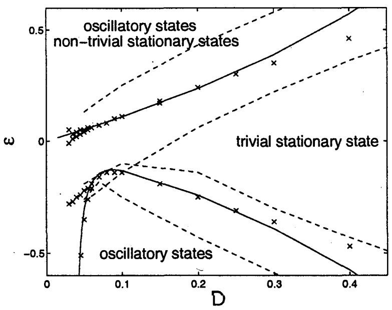

FIGURE 1. Phase diagram of the Langevin model (crosses), the Gaussian approximation (dashed lines) and the dichotomous theory (solid lines). Double lines indicate hysteretic transitions among phases.

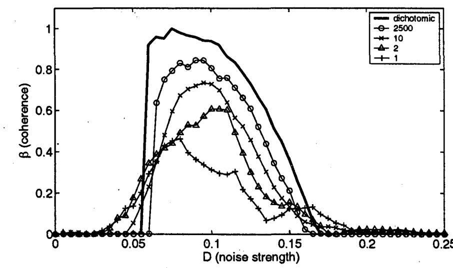

FIGURE 2. The normalized coherence β of the oscillatory state at $\varepsilon = -0.2$. as a function of the noise strength D, for networks of different size N as well as for the dichotomic theory. The coherence is given through $\beta = H\omega_p/\delta\omega$, where H is the hight of the dominant spectral peak at ω_p and $\delta\omega$ is the half-width of the peak.

mean field reads

$$\dot{X} = p_{12} - p_{21} - (p_{21} + p_{12})X, \tag{3}$$

where the hopping probabilities are given by the Kramers transition rate, which in the limit of small noise and small coupling strength reads (cf. [2])

$$p_{12,21} = \frac{\sqrt{2 \mp 3\varepsilon X(t-\tau)}}{2\pi} \exp\left(-\frac{1 \mp 4\varepsilon X(t-\tau)}{4D}\right). \tag{4}$$

PHASE DIAGRAM

A numerical study of the Langevin model (Eq. 1) shows that the system undergoes ordering transitions and demonstrates multistability. That is, for a strong enough positive coupling the system exhibits a non-zero stationary mean field and a variety of stable oscillatory states are accessible for positive and negative feedback. While the transition to the non-zero stationary mean field is second order (continuous), the type of the oscillatory transitions depends on the system parameters and can be first order (discontinuous), associated with hysteretic behavior, or second order.

A linear stability analysis of Eq. (3) yields the critical coupling strengths ε and the frequencies ω of the accessible oscillatory states.

A comparison of the Langevin model with the Gaussian approximation and the dichotomous theory shows that while near the transition points the system dynamics is strongly non-Gaussian, it is in this regime well described by a two-state model (see Fig. 1), which allows for a complete analysis of the bifurcations of the trivial equilibrium.

COHERENCE RESONANCE

The considered system exhibits the phenomena of coherence resonance and array-enhanced resonance (see Fig. 2).

Both, Kramers random switching frequency p (see 4) and the frequency of the oscillatory states ω, resulting from the coupling with the time-delayed mean field, depend on the noise strength, i.e., $p = p(D)$ and $\omega = \omega(D)$. Thus, the noise can be tuned so that the random hoping between the potential wells of the bistable oscillators is synchronized with the periodic modulation of the mean-field. This statistical synchronization takes place when $\omega = \pi p$, where the coherence of the oscillatory states becomes maximal.

For the $N = 1$ case the coherence resonance was observed by Tsimring and Pikovsky [2]. We observe that the resonance phenomenon not only persists in globally coupled networks with large N, but is enhanced, a property which was found in other systems and is sometimes referred to as array-enhanced resonance [3].

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