Dynamics of an Ensemble of Noisy Bistable Elements with Global Time Delayed Coupling

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The dynamics of an ensemble of bistable elements with global time-delayed coupling under the influence of noise is studied analytically and numerically. Depending on the noise level, the system undergoes ordering transitions and demonstrates multistability. That is, for a strong enough positive feedback it exhibits a nonzero stationary mean-field ,and a variety of stable oscillatory mean-field states are accessible for positive and negative feedback. The regularity of the oscillatory states is maximal for a certain noise level; i.e., the system demonstrates coherence resonance. While away from the transition points the system dynamics is well described by a Gaussian approximation, near the bifurcation points a description in terms of a dichotomous theory is more adequate.

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Noise induced hopping events in bistable or multistable systems form the basis of many interesting phenomena observed in physics, biology, chemistry, as well as social science. The study of such rate processes has thus been a subject of great interest, and various techniques such as Langevin, Fokker-Planck, and master equations have been used to describe the dynamics of stochastic systems [1–3].

In this Letter we generalize a well-studied stochastic model consisting of an ensemble of interconnected noise driven bistable oscillators by introducing time-delayed couplings. Such an extension is important since it has been realized that time delays are ubiquitous in nature and often change fundamentally the dynamics of the system [4–6].

The dynamics of the network is studied numerically by using Langevin equations and analytically by two complementary mean-field descriptions which are derived from the corresponding Fokker-Planck and master equations, respectively.

We assume that the bistable elements are highly interconnected, so that the connectivity can be approximated by a global all-to-all coupling. In such a system a bistable element may, for instance, model the basic properties of a neuron that can be in either a firing or a nonfiring state, a person that can opt between two choices by means of individual voting, or a gene that is either expressed or nonexpressed. The globally coupled system may then represent a highly interconnected neural network [7], a social group in which the individual voting behavior is influenced by opinion polls [8], or a genetic regulatory network [9], respectively. Other examples are catalytic surface reactions [10] and allosteric enzymic reactions [11].

The properties of globally coupled elements have been a subject of many studies [2,3,12,13]. In particular, Desai and Zwanzig [2] studied the synchronization of thermally activated bistable elements with instantaneous coupling and found an exact mean-field solution in the thermodynamic limit $N \to \infty$, where N is the number of elements in the network. This system exhibits a second order phase transition to an ordered state with nonzero stationary mean field. The effect of a time-delayed coupling has been studied by Yeung and Strogatz [13] for a globally coupled network of periodic phase oscillators, and Tsimring and Pikovsky [14] investigated the dynamics of a single bistable element driven by noise and time-delayed feedback.

Here, as in [13], it is assumed that the time delays between the bistable elements are uniform. Such an approximation is justified in certain neural networks, whose time delays are remarkably constant, as suggested by recent findings [15]. Similarly, for certain regulatory genetic networks the response time lag is determined by a single time constant [16].

Our prototype system for the study of rate processes in extended systems consists of N Langevin equations, each describing the overdamped noise driven motion of a particle in a bistable potential $V = -x^2/2 + x^4/4$, whose symmetry is distorted by a global coupling to the time-delayed mean field $X(t-\tau) = N^{-1}\sum_{i=1}^{N} x_i(t-\tau)$,

$$\dot{x}_i(t) = x_i(t) - x_i(t)^3 + \varepsilon X(t-\tau) + \sqrt{2D}\xi(t), \quad (1)$$

where τ is the time delay, ε is the coupling strength of the feedback, and D denotes the variance of the Gaussian fluctuations $\xi(t)$, which are mutually independent and uncorrelated $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$.

We first study system (1) numerically. The simulations are carried out using a fixed-step fifth-order Runge-Kutta method with linear interpolation for the evaluations at intermediate steps required for the delayed variables. If not otherwise stated, the time step and the number of elements are $\Delta t = 0.01 - 0.05$ and $N = 2500$, respectively.

For $\varepsilon = 0$, the elements are decoupled from each other. They jump from one potential well to the other randomly and independently of each other. Therefore, in this case the mean field $X = 0$. For small $|\varepsilon|$, the mean field remains zero. At a certain $\varepsilon = \varepsilon_{st} > 0$, which depends on the noise intensity D, but is independent of the time delay, the system undergoes a second order (continuous) phase transition and adopts a nonzero stationary mean field.

For a negative feedback, a transition to a periodically oscillating mean-field solution is observed at a certain $\varepsilon = \varepsilon_{osc-} < 0$. Here and for the rest of this Letter a $(-/+)$ index means that the corresponding value is associated with a negative/positive feedback.

Above a certain noise level D_H the transition at ε_{osc-} is second order as well. However, for $D < D_H$ the system exhibits a first order (discontinuous) transition associated with hysteretic behavior. The critical noise strength D_H depends on the time delay and is $D_H = 0.07$ for $\tau = 100$.

For large time delays $\tau \gg \tau_K$ (τ_K is the inverse Kramers escape rate from one well into the other), depending on the initial state the system adopts one of the many accessible oscillatory states featuring different periods. Even for a positive feedback, besides the stationary solution several oscillatory states with periods $T \lesssim \tau$ are observed for $\varepsilon > \varepsilon_{osc+} \gtrsim \varepsilon_{st}$. If the feedback is negative, the system has only oscillatory nontrivial solutions. The observed periods are $T \lesssim 2\tau$ for $\varepsilon < \varepsilon_{osc-}$.

The simulations show that for a negative feedback all oscillating states have a vanishing time-averaged mean field $\langle X \rangle_t$. However, for a positive feedback besides symmetric periodic solutions with $\langle X \rangle_t = 0$, states with significant nonzero temporal mean are possible.

In order to theoretically study the dynamical properties of a globally coupled set of noisy bistable elements (with no time delay), Desai and Zwanzig [2] derived a hierarchy of equations for the cumulant moments of the distribution function from the multidimensional Fokker-Planck equation for the joint probability distribution for all elements. For large noise intensities, when the statistics of individual elements are approximately Gaussian, this hierarchy can be truncated. Applying this approach to our system yields the following set of equations for the mean field X and the variance $M = N^{-1} \sum (x_i - X)^2$:

$$\dot{X}(t) = X(t) - X^3(t) - 3X(t)M(t) + \varepsilon X(t - \tau),$$
$$\tfrac{1}{2}\dot{M}(t) = M(t) - 3X^2(t)M(t) - 3M^2(t) + D. \qquad (2)$$

To compare the predictions of the Gaussian approximation (2) with the original Langevin model (1) we calculate the power of the main peak P_{peak} in the power spectrum of $X(t)$. It is proportional to the amplitude of the mean-field oscillations and can thus be used to analyze the Hopf bifurcation, which describes the transition to the oscillating mean-field states. The pitchfork bifurcation to the nonzero stationary mean field for a large enough positive coupling is characterized by the dependence of the temporal mean of the mean field $\langle X \rangle_t$ on the system parameters.

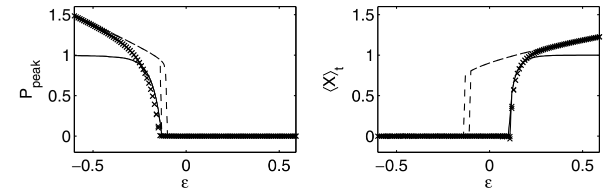

FIG. 1. The peak power P_{peak} and $\langle X \rangle_t$ as a function of the coupling strength for the Langevin model (crosses), the Gaussian approximation (dashed line), where the double line indicates hysteretic behavior, and the dichotomous theory (solid line). The constant noise strength is $D = 0.1$ and the time delay is $\tau = 100$. For $X = 0$ and $D = 0.1$ the Kramers time is $\tau_K = 54.1$.

For $\tau = 100$ the peak power P_{peak} and the temporal mean $\langle X \rangle_t$ resulting from the Gaussian approximation and the Langevin model are shown in Fig. 1 as a function of the coupling strength ε. The phase diagrams of these models are shown in Fig. 2 in the (D, ε)-parameter plane.

Figure 1 shows that away from the bifurcation points the Gaussian approximation describes the Langevin dynamics correctly. However, near the transition points the Langevin dynamics is strongly non-Gaussian even for large noise temperatures. For instance, the Gaussian approximation predicts that both bifurcations are subcritical for the entire noise range $D = 0.03$–1.0 considered in this study (see Fig. 2), while in the original Langevin model the bifurcations are subcritical only for $D < 0.07$.

Including higher-order cumulant equations leads only to a very slow convergence towards the true solution of the Langevin model. Thus, in order to describe the

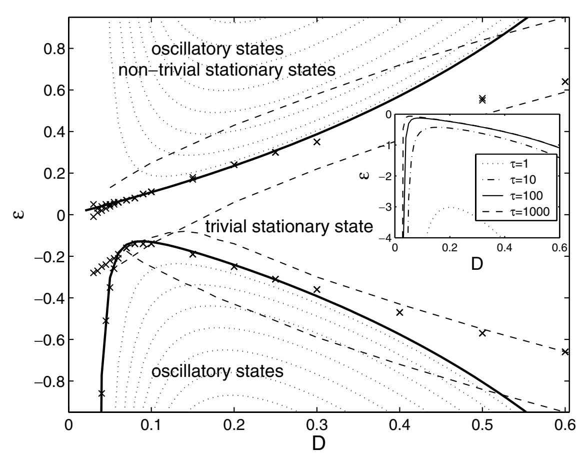

FIG. 2. Phase diagram for $\tau = 100$ of the Langevin model (crosses), the Gaussian approximation (dashed lines), and the dichotomous theory (solid lines and dotted lines). The solid line and the dotted line, respectively, depict the primary solution and the higher order solutions of Eq. (7) and (8). Phases separated by double lines indicate hysteretic behavior. The inset shows how the Hopf bifurcation line $\varepsilon^1_{osc-}(D)$ varies with the time delay τ. For $X = 0$ and $D < 0.3$ the Kramers time is $\tau_K > 10$.

behavior of the system near the bifurcation points, we apply a complementary *dichotomous* approximation, which is valid in the limit of small noise, where the characteristic Kramers transition time $\tau_K \gg 1$. In the dichotomous approximation intrawell fluctuations of x_i are neglected. Thus, each bistable element can be replaced by a discrete two-state system which can take only the values $x_{1,2} = \pm 1$. Then the collective dynamics of the entire system is described by the master equations for the occupation probabilities of these states $n_{1,2}$. This approach has been successfully used in studies of stochastic resonance and coherence resonance (e.g., [3,14,17,18]). For instance, using this approach Jung *et al.* [3] found stationary mean-field solutions in a globally coupled, time delay free network of bistable elements.

The dynamics of a single element is determined by the hopping rates p_{12} and p_{21}, i.e., by the probabilities to hop over the potential barrier from x_1 to x_2 and from x_2 to x_1, respectively. In a globally coupled system, $n_{1,2}$ and $p_{12,21}$ are identical for all elements. Then the master equations for the occupation probabilities read

$$\dot{n}_1 = -p_{12}n_1 + p_{21}n_2, \qquad \dot{n}_2 = p_{12}n_1 - p_{21}n_2. \quad (3)$$

In the dichotomous approximation the mean field $X = x_1n_1 + x_2n_2 = n_2 - n_1$, and making use of the probability conservation $n_1 + n_2 = 1$, we obtain the equation for the mean field

$$\dot{X}(t) = p_{12} - p_{21} - (p_{21} + p_{12})X(t). \quad (4)$$

The hopping probabilities $p_{12,21}$ are given by the Kramers transition rate [19] for the instantaneous potential well, which in the limit of small noise D and coupling strength ε reads (cf. [14])

$$p_{12,21} = \frac{\sqrt{2 \mp 3\varepsilon X(t-\tau)}}{2\pi} \exp\left(-\frac{1 \mp 4\varepsilon X(t-\tau)}{4D}\right). \quad (5)$$

A linear stability analysis of Eq. (4) near the trivial state $X = 0$ yields the transcendent equation for the complex eigenvalue λ,

$$\lambda = \frac{\sqrt{2}}{\pi} e^{-1/4D}\left(\frac{\varepsilon(4-3D)}{4D} e^{-\lambda\tau} - 1\right). \quad (6)$$

For a positive coupling this equation always has a real solution. At a certain critical coupling $\varepsilon_{st} = 4D/(4 - 3D)$ the eigenvalue becomes positive indicating the pitchfork bifurcation observed in the Langevin system (1). Besides this real solution, Eq. (6) possesses an infinite number of complex solutions corresponding to oscillating mean fields. However, only a finite number of them corresponds to unstable modes at finite τ and ε as we will see below. The critical values ε of the corresponding instabilities are found by substituting $\lambda = \mu + i\omega$ into Eq. (6), separating real and imaginary parts and setting $\mu = 0$:

$$\omega\tau = -\frac{\sqrt{2}}{\pi} \exp(-1/4D)\tau \tan\omega\tau, \quad (7)$$

$$\varepsilon_{osc} = \frac{\varepsilon_{st}}{\cos\omega\tau}. \quad (8)$$

This set of equations has a multiplicity of solutions, indicating that multistability occurs in the globally coupled system beyond a certain coupling strength. For finite time delays and positive coupling, besides the stationary solution, several oscillatory states with periods T_k close to but slightly larger than τ/k are observed for $\varepsilon > \varepsilon^k_{osc+}$ $\{k = 1, 2, \ldots\}$, where the transition points are ordered as follows, $0 < \varepsilon_{st} < \varepsilon^1_{osc+} < \varepsilon^2_{osc+} \ldots$. If the feedback is negative, the system has oscillatory solutions with periods T_l close to but slightly larger than $2\tau/(2l+1)$ for $\varepsilon < \varepsilon^l_{osc-}$ $\{l = 0, 1, \ldots\}$, where $0 > \varepsilon^0_{osc-} > \varepsilon^1_{osc-} \ldots$.

Let us now discuss the bifurcation properties in the limit of large and small time delays as well as vanishing noise and compare them with those of a single oscillator system. The critical coupling ε_{st} of the pitchfork bifurcation is time delay independent and goes to zero for vanishing noise. However, the critical coupling of the Hopf bifurcation depends on the time delay (see Fig. 2, inset). As the time delay increases, the maximum of the primary Hopf bifurcation line ε^1_{osc-} approaches the origin in the (ε, D) plane meaning that oscillations may occur at an arbitrary small feedback strength for the properly tuned noise level. This should be contrasted to the dynamics of a single noise-free oscillator with time-delayed feedback that exhibits oscillations only at strong negative feedback ($\epsilon < -1$). For very small time delays $\tau \to 0$, the critical coupling strength $\varepsilon^{l,k}_{osc\mp} \to \mp\infty$.

In order to compare the predictions of the dichotomous model with the Langevin dynamics, the peak power P_{peak} and the temporal mean $\langle X \rangle_t$, resulting from the dichotomous theory, are also plotted in Fig. 1. The phase diagram for the dichotomous theory is shown in Fig. 2.

Figures 1 and 2 show that the dichotomous theory agrees with the Langevin dynamics quite well for small noise in the range $D \approx 0.07$–0.3 in the neighborhood of the bifurcation points. The theory also correctly describes the bifurcation type. Indeed, the dichotomous theory predicts accurately the noise strength D_H ($= 0.07$ for $\tau = 100$) at which the Hopf bifurcation changes from supercritical to subcritical. However, for very small D the Kramers time becomes very large, and the accuracy of numerics becomes insufficient for a comparison with the theory.

Let us point out that the system studied in this Letter exhibits the phenomena of coherence resonance (e.g., [20]) and array-enhanced resonance. Since both Kramers random switching rate p [see Eq. (5)] and the frequency of the oscillatory states $f = \omega/(2\pi)$ [see Eq. (7)] depend on the noise strength, i.e., $p = p(D)$

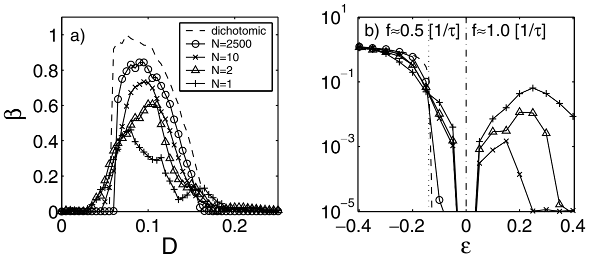

FIG. 3. The normalized coherence of the oscillatory states β for systems of different size N with $\tau = 100$. (a) β as a function of the noise strength D for $\varepsilon = 0.2$. (b) β as a function of the coupling strength ε for $D = 0.1$. For $\varepsilon < 0$ and $\varepsilon > 0$ the spectral peak frequency is $f_p \approx 0.5(1/\tau)$ and $f_p \approx 1.0(1/\tau)$, respectively. The dotted vertical line depicts $\varepsilon^1_{\mathrm{osc}-}$.

and $f = f(D)$, the noise can be tuned so that the random hopping between the potential wells of the bistable oscillators synchronizes with the periodic modulation of the mean field. This statistical synchronization takes place when $f = p/2$ [18], where the regularity of the oscillatory state becomes maximal. Here, this regularity is quantified by $\beta = Hf_p/\Delta f$, where H is the height of the spectral peak at f_p and Δf is the half width of the peak. The coherence measure β as a function of the noise strength is shown in Fig. 3(a). We observe that the regularity of the oscillatory states increases with increasing N, a property which was reported for other systems and is sometimes referred to as array-enhanced resonance [21]. Interestingly, the enhancement of the temporal regularity with increasing system size is observed only for macroscopic mean-field oscillations, while the inverse holds for "subcritical coherence." That is, the coherence observed in the power spectra of subcritical mean-field fluctuations (i.e., for $|\epsilon| < |\epsilon_{\mathrm{osc}\pm}|$) decays inversely proportional to the number of elements in the network, and becomes negligible for $N > 10$. This is shown in Fig. 3(b). Qualitatively, the same dependency on the system size is found if the delayed average does not include the delayed element itself, i.e., the element x_i is coupled to $X_i(t-\tau) = \sum_{j=1, j\neq i}^{N-1} x_j$.

In summary, we have shown that a network of noisy bistable elements with global time-delayed coupling possesses a multiplicity of stable oscillatory states for both positive and negative feedback in addition to a nonzero stationary mean field for a strong enough positive feedback which also occurs in a nondelayed system. These novel oscillatory states have a maximum regularity for a certain noise strength. The bifurcations of the trivial equilibrium are well described by the dichotomous theory in the limit of small noise and coupling strength. Far away from the bifurcation points the mean-field properties of the system are well described by the Gaussian approximation. However, the quantitative theory for the large noise strength near the bifurcation points is still lacking. In this Letter the effect of uniform time delays on the dynamics of a globally coupled network of bistable elements has been studied. However, many real networks have sparse coupling and nonuniform time delays. These properties should thus be included in a more general description. Preliminary results of our simulations with nonuniform time delays suggest that the bifurcation properties do qualitatively not change for a wide range of Gaussian distributed time delays, which substantiates the generic nature of the model considered here. This issue will be discussed in detail in an upcoming article.

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